Quantization of Interacting Non-Relativistic Open Strings using Extended Objects

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Abstract

Non-relativistic charged open strings coupled with Abelian gauge fields are quantized in a geometric representation that generalizes the Loop Representation. The model comprises open-strings interacting through a Kalb-Ramond field in four dimensions. It is shown that a consistent geometric-representation can be built using a scheme of "surfaces and lines of Faraday", provided that the coupling constant (the "charge" of the string) is quantized.

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I. INTRODUCTION

In this paper we consider theories of interacting non-relativistic strings, and quantize them in a representation that uses extended geometrical objets (paths and surfaces), that generalize the usual Loop Representation (LR) [1]. Special attention is devoted to the geometric quantization of interacting open strings. The surface representation was considered years ago to study the free-field case [2, 3], but it has to be adapted to include the particularities that the coupling with the string demands. To obtain a proper understanding of this theory we first consider the model of closed strings in self-interaction by means of an Abelian Kalb-Ramond field [4]. This model can be seen as a generalization of the theory of charged non-relativistic point particles, in electromagnetic interaction, quantized within the LR, exposed in reference [5], where it was found that the charge must be quantized in order to the LR formulation (Faraday's lines scheme) of the model be consistent. This result agrees with those obtained in previous developments [6, 7]. For both theories (interaction of closed and open strings) we find that the coupling constant of the string (let us say, the Kalb-Ramond "charge" of the string) must be also quantized, if the geometric representation adapted to the model is going to be plausible. On other hand, we should be aware that when we couple extended material objects to fields, the theory presents certain subtleties regarding its quantization, and so does the appropriate geometric representation.

The next section is devoted to the study of a geometric open path-surface representation for non-relativistic "charged" strings interacting by means of the Kalb-Ramond field. We discuss first the case of closed strings (subsection II A) treated previously in reference [8] and in subsection II B we consider the more interesting open string model, in a geometric representation. Final remarks are left to the last section.

II. NON-RELATIVISTIC INTERACTING "CHARGED" STRINGS

A. Closed string interaction

Let us start our discussion by introducing the action [8]

$$S = \frac{1}{12g^2}\int H^{\mu\nu\lambda}H_{\mu\nu\lambda}d^4x + \frac{\alpha}{2}\int dt \int d\sigma \left[(\dot{z}^i)^2 - (z'^i)^2)\right] + \frac{1}{2}\int d^4x J^{\mu\nu}B_{\mu\nu},$$

where the Kalb-Ramond antisymmetric potential and field strength, $B_{\mu\nu}$ and $H_{\mu\nu\lambda}$, respectively, are related by $H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]} = \partial_\mu B_{\nu\lambda} + \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\nu}$. Also we have a contribution corresponding to the free non-relativistic closed string, whose world sheet spatial coordinates $z^i(t,\sigma)$ are given in terms of the time t and the parameter σ along the string. The string tension α has units of $mass^2$ and g is a parameter with units of $mass$. The string-field interaction term is given by means of the current $J^{\mu\nu}(\vec{x},t) = \phi \int dt \int d\sigma \, [\dot{z}^\mu z'^\nu - \dot{z}^\nu z'^\mu] \, \delta^{(4)}(x-z)$. Here ϕ is the dimensionless coupling constant (analog to the charge in the case of particles), and we indicate with dots and primes partial derivation with respect to the time parameter t and σ, respectively.

We are interested in the Dirac quantization scheme of the theory so we define the conjugate momenta associated to the fields, B_{ij}, and string variables, z^i, as

$$\Pi^{ij} = \frac{1}{2g^2}\left(\dot{B}_{ij} + \partial_j B_{0i} - \partial_i B_{0j}\right), \qquad P_i = \alpha \dot{z}^i + \phi B_{ij} z'^j, \tag{1}$$

and directly obtain the Hamiltonian performing a Legendre transformation in the variables B_{ij} and z^i,

$$H = \int d^3x \left[g^2 \Pi^{ij}\Pi^{ij} + \frac{1}{12g^2} H_{ijk}H_{ijk}\right] + \int d\sigma \frac{\alpha}{2}\left[\frac{1}{\alpha^2}\left(P_i - \phi B_{ij}(z) z'^j\right)^2 + (z'^i)^2\right] + \int d^3x B_{0i}\chi^i. \tag{2}$$

B_{i0} are not dynamical variables, they appear as Lagrange multipliers enforcing the first class constraints

$$\chi^i(x) \equiv -\rho^i(x) - 2\partial_j \Pi^{ji}(x) = 0, \tag{3}$$

where $\rho^i(x) \equiv \phi \int d\sigma z'^i \delta^{(3)}(\vec{x} - \vec{z})$ is the "charge density" of the string. The preservation of the above constraints can be done using the defined canonical commutators algebra of the operators involved, whose non-vanishing conmutators are given by

$$\left[z^i(\sigma), P_j(\sigma')\right] = i\delta^i_j \delta(\sigma - \sigma') \qquad \left[B_{ij}(\vec{x}), \Pi^{kl}(\vec{y})\right] = i\frac{1}{2}\left(\delta^k_i \delta^l_j - \delta^l_i \delta^k_j\right)\delta^{(3)}(\vec{x} - \vec{y}), \tag{4}$$

This preservation does not produce new constraints.

Now, in order to solve relation (3), we introduce a geometric representation based on extended objects: an "open-surface representation", related with the LR formulated by Gambini and Trías [1] (and with a geometrical formulation based on closed surfaces [2, 3]). Consider the space of piecewise smooth oriented surfaces (for our purposes) in R^3. A typical element of this space, let us say Σ, will be the union of several surfaces, perhaps some

of them being closed. In the space of smoth oriented surfaces Σ we define equivalence classes of surfaces that share the same "form factor" $T^{ij}(x,\Sigma) = \int d\Sigma_y^{ij}\, \delta^{(3)}(\vec{x}-\vec{y})$, where $d\Sigma_y^{ij} = (\frac{\partial y^i}{\partial s}\frac{\partial y^j}{\partial r} - \frac{\partial y^i}{\partial r}\frac{\partial y^j}{\partial s})dsdr$ is the surface element and s, r are the parametrization variables. All the features of the "open surfaces space", are more or less immediate generalizations of aspects already present in the Abelian path space [1, 8, 9, 14]. Our Hilbert space is composed by functionals $\Psi(\Sigma)$ depending on equivalence classes Σ. We need to introduce the surface derivative $\delta_{ij}(x)$ defined by,

$$\Psi(\delta\Sigma \cdot \Sigma) - \Psi(\Sigma) = \sigma^{ij}\delta_{ij}(x)\Psi(\Sigma), \tag{5}$$

that measures the response of $\Psi(\Sigma)$ when an element of surface whose infinitesimal area $\sigma_{ij} = u^i v^j - v^j u^i$, generated by the infinitesimal vectors $\vec{u}$ and $\vec{v}$, is attached to Σ at the point x [2, 3, 8].

It can be seen that the fundamental commutator associated to equation (4) can be realized on surface-dependent functionals if one sets

$$\hat{\Pi}^{ij}(\vec{x}) \longrightarrow \frac{1}{2}T^{ij}(\vec{x},\Sigma), \qquad \hat{B}_{ij}(\vec{x}) \longrightarrow 2i\delta_{ij}(\vec{x}), \tag{6}$$

$$\hat{z}^i(\sigma) \longrightarrow z^i(\sigma), \qquad \hat{P}^i(\sigma) \longrightarrow -i\frac{\delta}{\delta z^i(\sigma)}, \tag{7}$$

and then the states of the interacting theory can be taken as functionals $\Psi[\Sigma, z(\sigma)]$, where the field is represented by the surface Σ and matter by means of the coordinates of the string world sheet. Of all of these functionals we must pick out those that belong to the kernel of the Gauss constraint (3), now written as

$$\left(\phi \int_{string} d\sigma z'^i \delta^{(3)}(\vec{x}-\vec{z}) - \int_{\partial\Sigma} d\sigma z'^i \delta^{(3)}(\vec{x}-\vec{z})\right)\Psi[\Sigma, z(\sigma)] \approx 0. \tag{8}$$

In the last equation we have used that $\partial_j T^{ji}(\vec{x},\Sigma) = -T^i(\vec{x},\partial\Sigma) = -\int_{\partial\Sigma} dz^i \delta^{(3)}(\vec{x}-\vec{z})$, with $\partial\Sigma$ being the boundary of the surface. At this point, we recall the geometrical setting that permits to solve the Gauss constraint in the theory of self-interacting non-relativistic particles coupled through Maxwell field [5]. In that case the physical space may be labelled by Faraday's lines and the scheme of quantization allows us to associate to every particle a bundle of lines emanating from or arriving to it, depending on the sign of the particle's charge. Then the charge must be quantized, since the number of open paths (that must be equal to the charge to which they are attached) has to be an integer [5]. In the present case,

we copy the interpretation and see that if the surface is such that its boundary coincides with the string, the constraint (8) reduces to $(\phi-1)\int_{string} d\sigma z'^j \delta^{(3)}(\vec{x}-\vec{z})=0$, and it is satisfied in general for $\phi=1$, so we can say in analogy that the surface emanates from the string. But it could happen that instead the boundary of the surface and the string have opposite orientations; in that case the constraint would be satisfied if $\phi=-1$, and we say that the surface "enters" or "arrives" at the string position. There is also the possibility that the surface could be composed by several layers (n of them) that start (or end) at the string. Equation (8) becomes $(\phi-n)\int_{string} d\sigma z'^j \delta^{(3)}(\vec{x}-\vec{z})=0$, and in this case the coupling constant ("charge" of the string) must obey $\phi=n$ (the sign of n depends on the fact that the surfaces may "emanate" from or "arrive" to the source). This is what we call a representation of "Faraday's surfaces" for the string-Kalb-Ramond system.

B. Open string interaction

Our starting point will be the action,

$$\mathcal{S}=\int dt\int d\sigma\frac{\alpha}{2}[(\dot{z}^i)^2-(z'^i)^2]+\int d^4x\left(\frac{1}{12g^2}H^{\mu\nu\lambda}H_{\mu\nu\lambda}-\frac{m^2}{4}a^{\mu\nu}a_{\mu\nu}+\frac{1}{2}J^{\mu\nu}B_{\mu\nu}+J^\mu A_\mu\right), \tag{9}$$

where we have defined the 2-form $a_{\mu\nu}=B_{\mu\nu}+F_{\mu\nu}$ with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ inspired in the Stückelberg gauge invariant version of the Proca model [14]. The vector field A_μ is dimensionless. The Kalb-Ramond antisymmetric potential and field strength are related as in (1). This action is invariant under the simultaneous gauge transformations,

$$B_{\mu\nu}\longrightarrow B_{\mu\nu}+\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu\,,\qquad A_\mu\longrightarrow A_\mu-\Lambda_\mu+\partial_\mu\lambda\,, \tag{10}$$

if the currents associated to the matter source (the body and the extremes of the string) satisfy the relation

$$\partial_\mu J^{\mu\nu}+J^\nu=0, \tag{11}$$

which emerges as a consequence of the gauge invariance and equations of motion. This implies that the current associated to particles placed in the extremes of the string is conserved $\partial_\nu J^\nu=0$, although the string-current $J^{\mu\nu}$ is not. This is a consequence of the string interaction term by means of two types of currents associated to matter, one that is associated

to the "body" and the other to the end points. This fact constitutes a desired aspect of the theory because permits the treatment of open-strings without losing gauge invariance. The equations of motion are given by

$$\partial_\mu H^{\mu\nu\lambda} + m^2 a^{\nu\lambda} - J^{\nu\lambda} = 0\,, \qquad m^2\partial_\nu a^{\nu\lambda} + J^{\lambda} = 0, \tag{12}$$

and they guarantee that (11) is satisfied.

Now, we proceed with Dirac quantization procedure. We take A_i, B_{ij} and z^i as dynamical variables and Π^i, Π^{ij} and P^i as their canonical conjugate momenta, respectively. The Hamiltonian density after a Legendre transformation in the dynamical variables results to be

$$\begin{aligned}\mathcal{H} \;=\; & g^2\Pi^{ij}\Pi^{ij} + \frac{\Pi^i\Pi^i}{2m^2} + \frac{1}{12g^2}H_{ijk}H_{ijk} + \frac{m^2}{4}a_{ij}a_{ij} - \frac{1}{2}J^{ij}B_{ij} - J^iA_i \\ & -B_{0i}(2\partial_j\Pi^{ji} + \Pi^i + J^{0i}) - A_0(\partial_i\Pi^i + J^0) + \int d\sigma\frac{\alpha}{2}\left[\frac{1}{\alpha^2}\left(P_i - \phi B_{ij}(z)z'^j\right)^2 + (z'^i)^2\right].\end{aligned} \tag{13}$$

Just as in the preceding subsection the fields variables A_0 and B_{i0} are treated as non-dynamical fields from the very beginning. They appear in $\mathcal{H}$ as Legendre multipliers enforcing the constraints

$$\Theta^i \equiv 2\partial_j\Pi^{ji} + \Pi^i + J^{0i} \approx 0\,, \qquad \Theta \equiv \partial_j\Pi^j + J^0 \approx 0, \tag{14}$$

where it can be seen that they are reducible because $\partial_i\Theta^i = \Theta$. Apart from the usual canonical Poisson algebra between the canonical conjugate variable, it is straight forward to infere that (see discussion in [14])

$$\{a_{ij}(\vec{x}), \Pi^{kl}(\vec{y})\} \;=\; \frac{1}{2}(\delta^k_i\delta^l_j - \delta^l_i\delta^k_j)\delta^{(3)}(\vec{x}-\vec{y}), \tag{15}$$

$$\{a_{ij}(\vec{x}), \Pi^{k}(\vec{y})\} \;=\; (\delta^k_j\partial_i - \delta^k_i\partial_j)\delta^{(3)}(\vec{x}-\vec{y}). \tag{16}$$

It can be shown that the preservation of the constraints (14) does not produce new ones. Furthermore, they result to be first class constraints that generate time independent gauge transformations on phase space.

To quantize, the canonical variables are promoted to operators obeying the commutators that result from the replacement $\{\ ,\ \} \longrightarrow -i[\ ,\ ]$. These operators have to be realized in a Hilbert space of physical states $|\Psi\rangle_{Phys}$, that obey the constraints (14) ($\Theta^i|\Psi\rangle_{Phys} = 0$).

At this point, we adapt a geometrical representation to the theory in terms of extended objects just as it was done in the case of self interacting closed-strings, but now considering (besides the surfaces) open paths γ associated to the fields A_i that mediate the interaction between the extreme points of the string. We prescribe,

$$\hat{\Pi}^{ij}(\vec{x}) \longrightarrow \frac{e}{2}T^{ij}(\vec{x},\Sigma)\,, \qquad \hat{\Pi}^{i}(\vec{x}) \longrightarrow eT^{i}(\vec{x},\gamma)\,, \qquad \hat{a}_{ij}(\vec{x}) \longrightarrow \frac{2i}{e}\delta_{ij}(\vec{x}), \tag{17}$$

where $T^i(\vec{x},\gamma)$ is the form factor that describes the open paths γ [1]. We can see, using $\partial_j T^{ji}(\vec{x},\Sigma) = -T^i(\vec{x},\partial\Sigma)$ and $\delta_{ij}(\vec{x})T^{lk}(\vec{y},\Sigma) = \frac{1}{2}(\delta_i^l\delta_j^k - \delta_i^k\delta_j^l)\delta^{(3)}(\vec{x}-\vec{y})$, that the fundamental commutators associated to equations (15) and (16) can be realized when they act over functionals depending on both surfaces and paths. Also we must pick out those functionals that belong to the kernel of the constraints (14). In this representation it can be expressed as

$$\begin{aligned}&\left(e\partial_i T^{ij}(\vec{x},\Sigma) + eT^j(\vec{x},\partial\Sigma') + J^{0j}\right)\psi(\Sigma,\gamma,\vec{z}(\sigma)) = \\ &\left(-eT^j(\vec{x},\partial\Sigma) + eT^j(\vec{x},\partial\Sigma') + \phi\int_C dz^j\delta^{(3)}(\vec{x}-\vec{z})\right)\psi(\Sigma,\gamma,\vec{z}(\sigma)) \approx 0.\end{aligned} \tag{18}$$

In (18), $\partial\Sigma'$ is the part of the border of the open surface that is drawn by the open-paths attached to the extreme points of the open strings. The rest of the surface border is completed by the strings, i.e., $\partial\Sigma = \partial\Sigma' + strings$.

III. DISCUSSION

Henceforth, we have the following interpretation: the states of the interacting theory of open strings can be taken as functionals $\Psi[\Sigma,\gamma,z(\sigma)]$ depending on surfaces and paths (i.e. the equivalence classes discussed above), and functions of the string variables $z(\sigma)$, that act as a source of the extended geometrical objects. The body of the string interacts using a surface, that can, depending on the orientation of the string (i.e., the coupling constant ϕ), "emanate" or "arrive" to it. The end points interact via the open paths (just as in the case of electromagnetic interaction for particles) that complete the part $\partial\Sigma'$ of the border of the surface that is not "glued" to the strings. Again, as in the closed string interaction with the Kalb-Ramond field the surface may consist of n layers attached to the string (depending on the value of the coupling constant ϕ), plus an arbitrary number of closed surfaces, since the latter do not contribute to the boundary of the surface.

Following references [5, 8], one could also consider the geometric representation of open strings interacting through topological terms, like a BF term in $3+1$ dimensions. This study, apart of being interesting regarding the solution of the constraints when the fields that provide the interaction have a topological character, has the particularity that the dependence of the wave-functionals on paths (or more generally, on the appropriate geometric objects that enter in the representation, like paths or surfaces) might be eliminated by means of an unitary transformation [5, 9]. In that case one could obtain a quantum mechanics of particles, or particles and strings (depending on the model), subjected to long range interactions [8] leading to anomalous statistic [10, 11, 12, 13]. This and other topics shall be the subject of future investigations.

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